

# On equifocal submanifolds with non-flat section in symmetric spaces of rank two

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## Abstract

In this paper, we show that there exists no equifocal submanifold with non-flat section in four irreducible simply connected symmetric spaces of rank two. Also, we show a fact for the sections of equifocal submanifolds with non-flat section in other irreducible simply connected symmetric spaces of rank two.

## 1. Introduction

A properly immersed complete submanifold  $M$  in a symmetric space  $G/K$  is called a *submanifold with parallel focal structure* if the following conditions hold:

(PF-i) the restricted normal holonomy group of  $M$  is trivial,

(PF-ii) if  $v$  is a parallel normal vector field on  $M$  such that  $v_{x_0}$  is a multiplicity  $k$  focal normal of  $M$  for some  $x_0 \in M$ , then  $v_x$  is a multiplicity  $k$  focal normal of  $M$  for all  $x \in M$ ,

(PF-iii) for each  $x \in M$ , there exists a properly embedded complete connected submanifold through  $x$  meeting all parallel submanifolds of  $M$  orthogonally.

The condition (PF-ii) is equivalent to the following condition:

(PF-ii') for each parallel unit normal vector field  $v$  of  $M$ , the set of all focal radii along the geodesic  $\gamma_{v_x}$  with  $\dot{\gamma}_{v_x}(0) = v_x$  is independent of the choice of  $x \in M$ .

This notion was introduced by Ewert ([E]). In [A], [AG] and [AT], this submanifold is simply called an *equifocal submanifold*. In this paper, we also shall use this name and assume that all equifocal submanifolds have trivial normal holonomy group. The submanifold as in (PF-iii) is called a *section of  $M$  through  $x$* , which is automatically totally geodesic. Here we note that, in 1995, Terng-Thorbergsson [TeTh] originally defined the notion of an equifocal submanifold as a compact

submanifold satisfying the conditions (PF-ii'), (PF-iii) and the condition that the normal holonomy group is trivial and that the section is flat. M.M. Alexandrino [A] has recently introduced the notion of a singular Riemannian foliation with section. Note that its regular leaves are equifocal. If,  $g_*^{-1}(T_{gK}^\perp M)$  is a Lie triple system of  $\mathfrak{p} := T_{eK}(G/K)$  for any  $gK \in M$ , then  $M$  is said to *have Lie triple systematic normal bundle*. Note that, under the condition (PF-i), the condition (PF-iii) is equivalent to the following condition:

(PF-iii')  $M$  has Lie triple systematic normal bundle.

In fact, (PF-iii) $\Rightarrow$ (PF-iii') is trivial and (PF-iii') $\Rightarrow$ (PF-iii) is shown as follows. If (PF-iii') holds, then it is shown by Proposition 2.2 of [HLO] that  $\exp^\perp(T_x^\perp M)$  meets all parallel submanifolds of  $M$  orthogonally for each  $x \in M$ , where  $\exp^\perp$  is the normal exponential map of  $M$ . Also, it is clear that  $\exp^\perp(T_x^\perp M)$  is properly embedded. Thus (PF-iii) follows. An isometric action of a compact Lie group  $H$  on a Riemannian manifold is said to be *polar* if there exists a properly embedded complete connected submanifold  $\Sigma$  meeting every principal orbits of the  $H$ -action orthogonally. The submanifold  $\Sigma$  is called a *section* of the action. If  $\Sigma$  is flat, then the action is said to be *hyperpolar*. Principal orbits of polar actions are equifocal submanifolds and those of hyperpolar actions are equifocal ones with flat section. Conversely, homogeneous equifocal submanifolds (resp. homogeneous equifocal ones with flat section) in the symmetric spaces occur as principal orbits of polar (resp. hyperpolar) actions on the spaces. For equifocal submanifolds with non-flat section, some open problems remain, for example the following.

**Open Problem 1.** *Does there exist no equifocal submanifold with non-flat section in an irreducible simply connected symmetric space of compact type and rank greater than one ?*

This includes the following open problem.

**Open Problem 2.** *Are all polar actions on irreducible simply connected symmetric spaces of compact type and rank greater than one hyperpolar ?*

L. Biliotti [Bi] gave the following partial answer for this problem.

*All polar actions on irreducible Hermitian symmetric spaces of compact type and rank greater than one are hyperpolar.*

He showed this fact by showing that all polar actions on a compact Kaehlerian manifold are coisotropic and that all coisotropic actions on irreducible Hermitian symmetric spaces of compact type are hyperpolar. See [Bi] about the definition of a coisotropic action. In 1985, Dadok [D] classified polar actions on spheres up to orbit equivalence. According to the classification, those actions are orbit equivalent to the restrictions to hyperspheres of the linear isotropy actions of irre-

ducible symmetric spaces. In 1999, Podestà and Thorbergsson [PoTh1] classified (non-hyperpolar) polar actions on simply connected rank one symmetric spaces of compact type other than spheres up to orbit equivalence. Kollross [Kol2] has recently showed that all polar actions on irreducible symmetric spaces  $G/K$ 's of type I (i.e.,  $G$  is irreducible) and rank greater than one are hyperpolar. Thus homogeneous equifocal submanifolds in irreducible symmetric space of type I are classified completely. All isoparametric submanifolds of codimension greater than one in a sphere are equifocal submanifolds with non-flat section. According to the homogeneity theorem by Thorbergsson ([Th]), if they are irreducible in a Euclidean space including the sphere as a hypersphere, then they are homogeneous and hence they occur as principal orbits of the linear isotropy actions of irreducible symmetric spaces of rank greater than two.

Let  $\Sigma$  be a totally geodesic rank one symmetric space in a symmetric space  $G/K$  of compact type and rank two. Without loss of generality, we may assume that  $eK \in \Sigma$ , where  $e$  is the identity element of  $G$ . Set  $\mathfrak{t} := T_{eK}\Sigma$ , which is a Lie triple system of  $\mathfrak{p} := T_{eK}(G/K) (\subset \mathfrak{g})$  ( $\mathfrak{g}$  : the Lie algebra of  $G$ ). Take  $v \in \mathfrak{t}$  and a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  containing  $v$ . It is shown that  $\mathfrak{t} \ominus \text{Span}\{v\}$  is orthogonal to  $\mathfrak{a}$ . Let  $\mathfrak{c}$  be a Weyl domain in  $\mathfrak{a}$  with  $v \in \bar{\mathfrak{c}}$ , where  $\bar{\mathfrak{c}}$  is the closure of  $\mathfrak{c}$ . If  $v \in \mathfrak{c}$ , then we say that  $\Sigma$  is *of principal type* and if  $v \in \partial\mathfrak{c}$ , then we say that it is *of singular type*. Note that this definition is independent of the choice of  $v$  and  $\mathfrak{a}$ . See Section 2 about maximal totally geodesic rank one symmetric spaces of principal type and singular type in irreducible simply connected symmetric spaces of compact type and rank two.

We [Koi3] showed the following fact:

*The sections of an equifocal submanifold with non-flat section in an irreducible symmetric space of compact type are isometric to a sphere or a real projective space.*

In this paper, we prove the following facts.

**Theorem A.** *There exists no equifocal submanifold with non-flat section in  $SU(3)/SO(3)$ ,  $SU(6)/Sp(3)$ ,  $E_6/F_4$  and  $SU(3)$ .*

**Theorem B.** *Let  $M$  be an equifocal submanifold with non-flat section in one of the following symmetric spaces:*

$$SU(2+q)/S(U(2) \times U(q)), SO(2+q)/SO(2) \times SO(q), SO(10)/U(5), \\ Sp(2+q)/Sp(2) \times Sp(q), E_6/Spin(10) \cdot U(1), G_2/SO(4), Sp(2), G_2.$$

*Then the sections of  $M$  are totally geodesic spheres (or real projective spaces) of singular type.*

## 2. Maximal totally geodesic rank one symmetric spaces

S. Klein [Kl1~4] has recently classified totally geodesic submanifolds in all irreducible simply connected symmetric spaces of rank two. In this section, according to his classifications, we give the list of maximal totally geodesic rank one symmetric spaces of principal type and singular type in irreducible simply connected symmetric spaces of compact type and rank two:

$$SU(3)/SO(3), SU(6)/Sp(3), SU(2+q)/S(U(2) \times SU(q)), \\ SO(2+q)/SO(2) \times SO(q), SO(10)/U(5), Sp(2+q)/Sp(2) \times Sp(q), \\ E_6/Spin(10) \cdot U(1), E_6/F_4, G_2/SO(4), SU(3), Sp(2), G_2,$$

where the maximality means that it is maximal among totally geodesic rank one symmetric spaces. Let  $G/K$  be an irreducible simply connected symmetric space of compact type and rank two and set  $\mathfrak{p} := T_{eK}(G/K)$ . Take a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ . Let  $\Delta$  be the root system with respect to  $\mathfrak{a}$  and  $\Delta_+$  be the positive root system under a lexicographical ordering of the dual space  $\mathfrak{a}^*$  of  $\mathfrak{a}$ . Also, let  $\Pi = \{\alpha_1, \alpha_2\} (\subset \Delta_+)$  be the simple root system of  $\Delta$ , where  $\alpha_2$  is the longer one of two elements of  $\Pi$  when  $\Delta$  is of  $(\mathfrak{b}_2)$  or  $(\mathfrak{g}_2)$ . A Weyl domain  $\mathfrak{c}$  in  $\mathfrak{a}$  is given by  $\mathfrak{c} = \{v \in \mathfrak{a} \mid \alpha_i(v) > 0 \ (i = 1, 2)\}$ . Let  $v_i$  ( $i = 1, 2$ ) be the unit vector of  $\alpha_i^{-1}(0)$  belonging to the closure  $\bar{\mathfrak{c}}$  of  $\mathfrak{c}$ . Set  $v_\theta := \cos \theta v_1 + \sin \theta v_2$  ( $0 \leq \theta \leq \theta_0$ ), where  $\theta_0 = \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}$  or  $\frac{\pi}{6}$  following to  $\Delta$  is of  $(\mathfrak{a}_2), (\mathfrak{b}_2), (\mathfrak{bc}_2)$  or  $(\mathfrak{g}_2)$  type. If  $\mathfrak{t} (\subset \mathfrak{p})$  is a Lie triple system and  $\text{Exp } \mathfrak{t}$  is a totally geodesic rank one symmetric space, then  $\mathfrak{t}$  is contained in the cone  $C(\text{Ad}(K) \cdot v_\theta)$  (in  $\mathfrak{p}$ ) over the orbit  $\text{Ad}(K) \cdot v_\theta$  of the  $s$ -representation associated with  $G/K$  through  $v_\theta$  for some  $\theta \in [0, \theta_0]$  (see Section 2 of [Kl4]), where  $\text{Exp}$  is the exponential map of  $G/K$  at  $eK$  and  $\text{Ad}$  is the adjoint representation of  $G$ . The angle  $\theta$  is called the *isotropy angle* of  $\mathfrak{t}$ . This terminology was originally used in [Kl4]. Here we note that  $\text{Exp } \mathfrak{t}$  is a totally geodesic rank one symmetric space of singular type if and only if the isotropy angle of  $\mathfrak{t}$  is equal to 0 or  $\theta_0$ .

First we consider the case of  $G/K = SU(3)/SO(3)$ . Then  $\Delta$  is of  $(\mathfrak{a}_2)$ -type. According to Table in Section 4.6 of [Kl4], for each  $\theta \in [0, \frac{\pi}{3}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $SU(3)/SO(3)$  are as in Table 1. Here we note that, for each  $\theta$ ,  $\Sigma_\theta$  is not unique.

$\theta$	$\Sigma_\theta$	Type
$\frac{\pi}{6}$	$S^2, \mathbb{R}P^2$	principal
$0, \frac{\pi}{3}$	$S^1$	singular
other	$S^1, \mathbb{R}$	principal

**Table 1.**

Next we consider the case of  $G/K = SU(6)/Sp(3)$ . Then  $\triangle$  is of  $(\mathfrak{a}_2)$ -type. According to Table in Section 4.4 of [K14], for each  $\theta \in [0, \frac{\pi}{3}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $SU(6)/Sp(3)$  are as in Table 2.

$\theta$	$\Sigma_\theta$	Type
$\frac{\pi}{6}$	$S^5, \mathbb{C}P^3, \mathbb{Q}P^2$	principal
$0, \frac{\pi}{3}$	$S^1$	singular
other	$S^1, \mathbb{R}$	principal

**Table 2.**

Next we consider the case of  $G/K = SU(2+q)/S(U(2) \times U(q))$  ( $q \geq 2$ ). Then  $\triangle$  is of  $(\mathfrak{b}_2)$ -type. According to Theorem 7.1 of [K12], for each  $\theta \in [0, \frac{\pi}{4}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $SU(2+q)/S(U(2) \times U(q))$  are as in Table 3.

$\theta$	$\Sigma_\theta$	Type
0	$\mathbb{C}P^q$	singular
$\arctan \frac{1}{3}$	$S^2$	principal
$\arctan \frac{1}{2}$	$\begin{cases} \mathbb{C}P^2 & (\text{when } q \geq 4) \\ \mathbb{R}P^2, S^4 & (\text{when } q = 3) \\ S^1 & (\text{when } q = 2) \end{cases}$	principal
$\frac{\pi}{4}$	$\mathbb{Q}P^{[\frac{q}{2}]}$	singular
other	$S^1, \mathbb{R}$	principal

**Table 3.**

Next we consider the case of  $G/K = SO(2+q)/SO(2) \times SO(q)$  ( $q \geq 3$ ). Then  $\triangle$  is of  $(\mathfrak{b}_2)$ -type. According to Theorem 4.1 and Proposition 3.7 of [K11], for each  $\theta \in [0, \frac{\pi}{4}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $SO(2+q)/SO(2) \times SO(q)$  are as in Table 4.

$\theta$	$\Sigma_\theta$	Type
0	$\mathbb{R}P^2, S^q$	singular
$\arctan \frac{1}{2}$	$S^2$	principal
$\frac{\pi}{4}$	$\mathbb{C}P^{[\frac{q}{2}]}$	singular
other	$S^1, \mathbb{R}$	principal

**Table 4.**

Next we consider the case of  $G/K = SO(10)/U(5)$ . Then  $\Delta$  is of  $(\mathfrak{b}_2)$ -type. According to Theorem 3.10 and Table in Section 3.5 of [Kl4], for each  $\theta \in [0, \frac{\pi}{4}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $SO(10)/U(5)$  are as in Table 5.

$\theta$	$\Sigma_\theta$	Type
0	$\mathbb{C}P^4$	singular
$\frac{\pi}{4}$	$S^6$	singular
other	$S^1, \mathbb{R}$	principal

**Table 5.**

Next we consider the case of  $G/K = Sp(2+q)/Sp(2) \times Sp(q)$  ( $q \geq 2$ ). Then  $\Delta$  is of  $(\mathfrak{b}_2)$ -type. According to Theorem 5.3 of [Kl2], for each  $\theta \in [0, \frac{\pi}{4}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $Sp(2+q)/Sp(2) \times Sp(q)$  are as in Table 6.

$\theta$	$\Sigma_\theta$	Type
0	$\mathbb{Q}P^q$	singular
$\arctan \frac{1}{3}$	$S^3$	principal
$\arctan \frac{1}{2}$	$\begin{cases} \mathbb{Q}P^2 & (\text{when } q \geq 5) \\ \mathbb{C}P^2, \mathbb{Q}P^1 & (\text{when } q = 4) \\ \mathbb{R}P^2, \mathbb{Q}P^1 & (\text{when } q = 3) \\ \mathbb{Q}P^1 & (\text{when } q = 2) \end{cases}$	principal
$\frac{\pi}{4}$	$S^5, \mathbb{Q}P^{[\frac{q}{2}]}$	singular
other	$S^1, \mathbb{R}$	principal

**Table 6.**

Next we consider the case of  $G/K = E_6/Spin(10) \cdot U(1)$ . Then  $\Delta$  is of  $(\mathfrak{b}_2)$ -type. According to Theorem 3.3 in Section 3.2 of [KL4] and Table in Section 3.3

of [KL4], for each  $\theta \in [0, \frac{\pi}{4}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $E_6/\text{Spin}(10) \cdot U(1)$  are as in Table 7.

$\theta$	$\Sigma_\theta$	Type
0	$\mathbb{C}P^5$	singular
$\frac{\pi}{4}$	$\mathbb{O}P^2$	singular
other	$S^1, \mathbb{R}$	principal

**Table 7.**

Next we consider the case of  $G/K = E_6/F_4$ . Then  $\Delta$  is of  $(\mathfrak{a}_2)$ -type. According to Theorem 4.2 and Table in Section 4.2 of [KL4] and Table in Section 4.3 of [KL4], for each  $\theta \in [0, \frac{\pi}{3}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $E_6/F_4$  are as in Table 8.

$\theta$	$\Sigma_\theta$	Type
$\frac{\pi}{6}$	$S^9, \mathbb{Q}P^3, \mathbb{O}P^2$	principal
$0, \frac{\pi}{3}$	$S^1$	singular
other	$S^1, \mathbb{R}$	principal

**Table 8.**

Next we consider the case of  $G/K = G_2/SO(4)$ . Then  $\Delta$  is of  $(\mathfrak{g}_2)$ -type. According to Theorem 5.2 and Table in Section 5.2 of [KL4], for each  $\theta \in [0, \frac{\pi}{6}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $G_2/SO(4)$  are as in Table 9.

$\theta$	$\Sigma_\theta$	Type
0	$S^3$	singular
$\arctan(\frac{1}{3\sqrt{3}})$	$S^3$	principal
$\frac{\pi}{6}$	$S^3, \mathbb{R}P^3, \mathbb{C}P^2$	singular
other	$S^1, \mathbb{R}$	principal

**Table 9.**

Next we consider the case of  $G/K = (SU(3) \times SU(3))/\Delta SU(3) = SU(3)$ . Then  $\Delta$  is of  $(\mathfrak{a}_2)$ -type. According to Theorem 4.6 and Table in Section 4.5 of [KL4], for each  $\theta \in [0, \frac{\pi}{3}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $SU(3)$  are as in Table 10.

$\theta$	$\Sigma_\theta$	Type
$\frac{\pi}{6}$	$S^3, \mathbb{R}P^3, \mathbb{C}P^2$	principal
$0, \frac{\pi}{3}$	$S^1$	singular
other	$S^1, \mathbb{R}$	principal

**Table 10.**

Next we consider the case of  $G/K = (Sp(2) \times Sp(2))/\Delta Sp(2) = Sp(2)$ . Then  $\Delta$  is of  $(\mathfrak{b}_2)$ -type. According to Theorem 3.8 and Table in Section 3.4 of [KL4], for each  $\theta \in [0, \frac{\pi}{4}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $Sp(2)$  are as in Table 11.

$\theta$	$\Sigma_\theta$	Type
$\arctan \frac{1}{3}$	$S^3$	principal
$\frac{\pi}{4}$	$S^3, \mathbb{Q}P^1$	singular
0	$S^1$	singular
other	$S^1, \mathbb{R}$	principal

**Table 11.**

Next we consider the case of  $G/K = (G_2 \times G_2)/\Delta G_2$ . Then  $\Delta$  is of  $(\mathfrak{g}_2)$ -type. According to Theorem 5.2 and Table in Section 5.2 of [KL4] and Table in Section 5.3 of [KL4], for each  $\theta \in [0, \frac{\pi}{6}]$ , maximal totally geodesic rank one symmetric spaces  $\Sigma_\theta$  having  $\theta$  as the isotropy angle in  $G_2$  are as in Table 12.

$\theta$	$\Sigma_\theta$	Type
0	$S^3$	singular
$\arctan \frac{1}{3\sqrt{3}}$	$S^3$	principal
$\frac{\pi}{6}$	$S^3, \mathbb{R}P^3, \mathbb{C}P^2$	singular
other	$S^1, \mathbb{R}$	principal

**Table 12.**

### 3. Proof of Theorems A and B

Let  $G/K$  be an irreducible simply connected symmetric space of compact type,  $\Delta$  be the root system of  $G/K$  with respect to a maximal abelian subspace



of  $\mathfrak{p} := T_{eK}(G/K) (\subset \mathfrak{g} := \text{Lie } G)$ ,  $M$  be an equifocal submanifold with non-flat section in  $G/K$  and  $\Sigma_x$  be the section of  $M$  through  $x$ . Without loss of generality, we may assume that  $G$  is simply connected and  $K$  is connected. Now we shall prepare some lemmas to prove Theorems A and B.

**Lemma 1.** *If  $\Sigma$  is a totally geodesic rank one symmetric space of principal type (resp. singular type), then a maximal totally geodesic rank one symmetric space containing  $\Sigma$  also is of principal type (resp. singular type).*

*Proof.* This fact is trivial from the definition of a totally geodesic rank one symmetric space of principal type (resp. singular type). q.e.d.

Next we prove the following lemma for maximal totally geodesic rank one symmetric spaces of principal type.

**Lemma 2.** *Let  $\Sigma_1$  and  $\Sigma_2$  be maximal totally geodesic spheres (or real projective spaces) of principal type. If  $\dim(\Sigma_1 \cap \Sigma_2) \geq 1$ , then we have  $\Sigma_1 = \Sigma_2$ .*

*Proof.* Without loss of generality, we may assume that  $eK \in \Sigma_1 \cap \Sigma_2$ . Set  $\mathfrak{t}_i := T_{eK}\Sigma_i$  ( $i = 1, 2$ ). Take  $v(\neq 0) \in \mathfrak{t}_1 \cap \mathfrak{t}_2$  and a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p} := T_{eK}(G/K)$  containing  $v$ . Since  $\Sigma_i$  ( $i = 1, 2$ ) are of principal type,  $v$  belongs to a Weyl domain  $\mathfrak{c}$  in  $\mathfrak{a}$ . Hence  $\mathfrak{a}$  is the only maximal abelian subspace of  $\mathfrak{p}$  containing  $v$ . According to the classification of totally geodesic submanifolds by S. Klein ([K1]~[K14]), it follows from this fact that  $\Sigma_1 = \Sigma_2$ . q.e.d.

Here we shall show that the fact similar to the statement of Lemma 2 does not hold for maximal totally geodesic spheres (or real projective spaces) of singular type. Let  $\Sigma$  be a maximal totally geodesic sphere (or real projective space) of singular type containing  $eK$ . Set  $\mathfrak{t} := T_{eK}\Sigma$ . Take  $v(\neq 0) \in \mathfrak{t}$  and a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  containing  $v$ . Since  $\Sigma$  is of singular type,  $v$  belongs to the boundary  $\partial\mathfrak{c}$  of a Weyl domain  $\mathfrak{c}$  in  $\mathfrak{a}$ . Let  $\Delta$  be the root system of  $G/K$  with respect to  $\mathfrak{a}$  and  $\Pi = \{\alpha_1, \alpha_2\}$  be a simple root system of  $\Delta$  satisfying  $\{w \in \mathfrak{a} \mid \alpha_i(w) > 0 \ (i = 1, 2)\} = \mathfrak{c}$  and  $\alpha_1(v) = 0$ . Then  $\text{Ad}(\exp Z_0)|_{\mathfrak{a}}$  gives the reflection in  $\mathfrak{a}$  with respect to  $\alpha_1^{-1}(0)$  for some  $Z_0 \in \mathfrak{f}_{\alpha_1} := \{Z \in \mathfrak{f} := \text{Lie } K \mid \text{ad}(a)^2(Z) = -\alpha_1(a)^2 Z \ (\forall a \in \mathfrak{a})\}$ . We define  $\Sigma_t$  ( $t \in \mathbb{R}$ ) by  $\Sigma_t := \text{Exp}(\text{Ad}(\exp tZ)(\mathfrak{t}))$ . For many  $\Sigma$ 's,  $\{\Sigma_t\}_{t \in \mathbb{R}}$  is a non-trivial smooth one-parameter family of maximal totally geodesic spheres (or real projective spaces) of singular type and  $\dim \left( \bigcap_{t \in \mathbb{R}} \Sigma_t \right) \geq 1$  holds because of  $v \in T_{eK} \left( \bigcap_{t \in \mathbb{R}} \Sigma_t \right)$ . Thus the fact similar to the statement of Lemma 2 does not

hold for maximal totally geodesic spheres (or real projective spaces) of singular type.

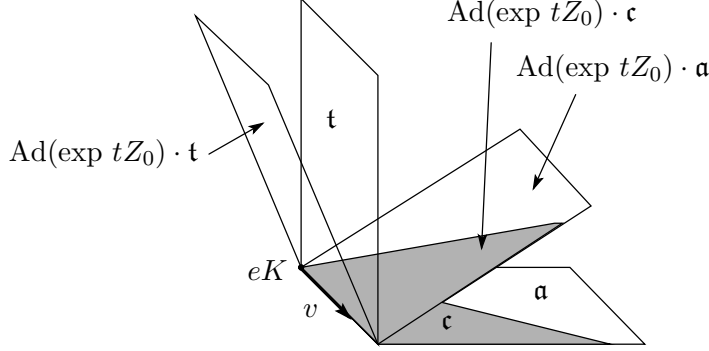


Fig. 1.

Let  $H^0([0, 1], \mathfrak{g})$  be the space of all  $L^2$ -integrable paths in  $\mathfrak{g} := \text{Lie } G$  (the Lie algebra of  $G$ ) having  $[0, 1]$  as the domain and  $H^1([0, 1], G)$  be the Hilbert Lie group of all  $H^1$ -paths in  $G$  having  $[0, 1]$  as the domain. Let  $\pi : G \rightarrow G/K$  be the natural projection and  $\phi : H^0([0, 1], \mathfrak{g}) \rightarrow G$  be the parallel transport map for  $G$ , which is defined by  $\phi(u) := g_u(1)$  for  $u \in H^0([0, 1], \mathfrak{g})$ , where  $g_u$  is the element of  $H^1([0, 1], G)$  satisfying  $g_u(0) = e$  and  $g_{u*}^{-1} g'_u = u$ . Let  $\widetilde{M} := (\pi \circ \phi)^{-1}(M)$ . Since  $G$  is simply connected and  $K$  is connected,  $\widetilde{M}$  are connected. Denote by  $A$  (resp.  $\widetilde{A}$ ) the shape tensor of  $M$  (resp.  $\widetilde{M}$ ) and by  $\nabla^\perp$  (resp.  $\widetilde{\nabla}^\perp$ ) the normal connection of  $M$  (resp.  $\widetilde{M}$ ). Without loss of generality, we may assume that  $eK \in M$ , where  $e$  is the identity element of  $G$ . Hence we have  $\hat{0} \in \widetilde{M}$ , where  $\hat{0}$  is the constant path at the zero element of  $\mathfrak{g}$ . Take  $v \in T_{\hat{0}}^\perp M$ . Take a maximal abelian subspace of  $\mathfrak{p}$  and a maximal abelian subalgebra  $\widetilde{\mathfrak{a}}$  containing  $\mathfrak{a}$ . Set  $\mathfrak{a}_{\mathfrak{f}} := \widetilde{\mathfrak{a}} \cap \mathfrak{f}$ , where  $\mathfrak{f}$  is the Lie algebra of  $K$ . Let  $\Delta$  be the root system of  $G/K$  with respect to  $\mathfrak{a}$ ,  $\Delta_+$  be the positive root system under a lexicographical ordering of  $\mathfrak{a}^*$  and  $\mathfrak{p}_\alpha$  be the root space for  $\alpha \in \Delta_+$ . Also, set  $\mathfrak{f}_\alpha := \text{ad}(a)\mathfrak{p}_\alpha$ , where  $a$  is the regular element of  $\mathfrak{a}$ . For convenience, we denote  $\mathfrak{a}$  by  $\mathfrak{p}_0$  and the centralizer  $\mathfrak{z}_{\mathfrak{f}}(\mathfrak{a})$  of  $\mathfrak{a}$  in  $\mathfrak{f}$  by  $\mathfrak{f}_0$ . For  $X \in \mathfrak{p}_\alpha$  ( $\alpha \in \Delta_+ \cup \{0\}$ ), we define  $X_{\mathfrak{f}}$  as the element of  $\mathfrak{f}$  such that  $\text{ad}(a)(X) = \alpha(a)X_{\mathfrak{f}}$  and  $\text{ad}(a)(X_{\mathfrak{f}}) = -\alpha(a)X$  for all  $a \in \mathfrak{a}$ . For  $X \in \mathfrak{p}_\alpha$ ,  $Y \in \widetilde{\mathfrak{a}}$  and  $k \in \mathbf{Z}$ , we define loop vectors  $l_{X,k}^i$ ,  $l_{X_{\mathfrak{f}},k}^i$  and  $l_{Y,k}^i \in H^0([0, 1], \mathfrak{g})$  ( $i = 1, 2$ ) by

$$\begin{aligned} l_{X,k}^1(t) &= l_{X_{\mathfrak{f}},k}^1(t) = X \cos(2k\pi t) - X_{\mathfrak{f}} \sin(2k\pi t), \\ l_{X,k}^2(t) &= l_{X_{\mathfrak{f}},k}^2(t) = X \sin(2k\pi t) + X_{\mathfrak{f}} \cos(2k\pi t), \\ l_{Y,k}^1(t) &= Y \cos(2k\pi t), \quad l_{Y,k}^2(t) = Y \sin(2k\pi t). \end{aligned}$$

For a general  $Z \in \mathfrak{g}$ , we define loop vectors  $l_{Z,k}^i \in H^0([0, 1], \mathfrak{g})$  ( $i = 1, 2, k \in \mathbf{Z}$ )

by

$$l_{Z,k}^i := l_{Z_0,k}^i + \sum_{\alpha \in \Delta_+^v} \left( l_{Z_{\mathfrak{p},\alpha},k}^i + l_{Z_{\mathfrak{f},\alpha},k}^i \right),$$

where  $Z = Z_0 + \sum_{\alpha \in \Delta_+^v} (Z_{\mathfrak{p},\alpha} + Z_{\mathfrak{f},\alpha})$  ( $Z_0 \in \tilde{\mathfrak{a}}$ ,  $Z_{\mathfrak{p},\alpha} \in \mathfrak{p}_\alpha$ ,  $Z_{\mathfrak{f},\alpha} \in \mathfrak{f}_\alpha := \{X_{\mathfrak{f}} \mid X \in \mathfrak{p}_\alpha\}$ ).

Denote by  $\hat{*}$  the constant path at  $* \in \mathfrak{g}$ . Note that  $\hat{*}$  is the horizontal lift of  $*$  ( $\in \mathfrak{g} = T_e G$ ) to  $\hat{0}$ . Then, according to Propositions 3.1 and 3.2 of [Koi2] and those proofs, we have the following relations.

**Lemma 3.** *Let  $X \in T_{eK}M \cap \mathfrak{p}_\alpha$  ( $\alpha \in \Delta_+ \cup \{0\}$ ). Then we have*

$$\begin{aligned} \tilde{A}_{\hat{v}} l_{X,k}^1 &= \frac{\alpha(v)}{2k\pi} (\hat{X} - l_{X,k}^1), \\ \tilde{A}_{\hat{v}} l_{X,k}^2 &= \frac{\alpha(v)}{2k\pi} (\hat{X}_{\mathfrak{f}} - l_{X,k}^2), \\ \tilde{A}_{\hat{v}} \hat{X} &= \widehat{A_v X} - \frac{\alpha(v)}{2} \hat{X}_{\mathfrak{f}} + \frac{\alpha(v)}{2\pi} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{1}{k} l_{X,k}^1, \\ \tilde{A}_{\hat{v}} \hat{X}_{\mathfrak{f}} &= -\frac{\alpha(v)}{2} \hat{X} + \frac{\alpha(v)}{2\pi} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{1}{k} l_{X,k}^2. \end{aligned}$$

**Lemma 4.** *Let  $w \in T_{eK}^\perp M \cap \mathfrak{p}_\alpha$  ( $\alpha \in \Delta_+ \cup \{0\}$ ). Then we have*

$$\begin{aligned} \tilde{A}_{\hat{v}} l_{w,k}^1 &= -\frac{\alpha(v)}{2k\pi} l_{w,k}^1, \\ \tilde{A}_{\hat{v}} l_{w,k}^2 &= \frac{\alpha(v)}{2k\pi} (\hat{w}_{\mathfrak{f}} - l_{w,k}^2), \\ \tilde{A}_{\hat{v}} \hat{w}_{\mathfrak{f}} &= \frac{\alpha(v)}{2\pi} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{1}{k} l_{w,k}^2. \end{aligned}$$

**Lemma 5.** *Let  $X \in \mathfrak{a}_{\mathfrak{f}}$ . Then we have*

$$\tilde{A}_{\hat{v}} l_{X,k}^i = \tilde{A}_{\hat{v}} \hat{X} = 0.$$

From these lemmas, the following relation follows directly.

**Lemma 6.** *The following relation holds:*

$$\begin{aligned} \text{Ker } \tilde{A}_v &= \text{Span}\{\hat{X} \mid X \in \text{Ker } A_v \cap \text{Ker } R(\cdot, v)v\} \oplus \text{Span}\{\hat{\eta} \mid \eta \in \mathfrak{z}_{\mathfrak{f}}(\text{Span}\{v\})\} \\ &\quad \oplus \text{Span}\{l_{Z,k}^i \mid Z \in \mathfrak{c}_{\mathfrak{g}}(\text{Span}\{v\}), i = 1, 2, k \in \mathbf{N} \setminus \{0\}\}. \end{aligned}$$

According to Lemma 1A.4 of [PoTh1], we see that, for any  $x(=gK) \in M$ , the focal set of  $(M, x)$  consists of finitely many totally geodesic hypersurfaces in the section  $\Sigma$  through  $x$ . Denote by  $\mathfrak{FH}_x$  the set of all totally geodesic hypersurfaces in  $\Sigma$  constructing the focal set of  $(M, x)$ .

By using Lemmas 1, 2 and 6, we shall prove Theorem A.

*Proof of Theorem A .* (The case of  $SU(3)/SO(3)$ ) Suppose that there exists an equifocal submanifold  $M$  with non-flat section in  $SU(3)/SO(3)$ . Without loss of generality, we may assume that  $eK \in M$ . Denote by  $\Sigma_x$  the section of  $M$  through  $x( \in M)$ . According to Table 1,  $\Sigma_x$  is a 2-dimensional maximal totally geodesic sphere (or real projective space) of the isotropy angle  $\frac{\pi}{6}$ , where the maximality means that it is maximal among totally geodesic spheres or totally geodesic real projective spaces. Take  $\sigma \in \mathfrak{FH}_{eK}$  and a focal normal vector field  $v$  of  $M$  such that  $p := \exp^\perp(v_{eK})$  belongs to  $\sigma$  but that it does not belong to members of  $\mathfrak{FH}_{eK}$  other than  $\sigma$ . Also, let  $F_\sigma$  be a focal submanifold corresponding to  $v$  (i.e.,  $F := \eta_v(M)$  ( $\eta_v$  : the end-point map for  $v$ )). Set  $L_\sigma := \eta_v^{-1}(p)$ , which is contained in  $\exp^\perp(T_p^\perp F)$ . Then we have  $\bigcup_{y \in L_\sigma} \Sigma_y = \exp^\perp(T_p^\perp F)$  and  $\Sigma_{y_1} \cap \Sigma_{y_2} = \sigma$  holds for any distinct points  $y_1$  and  $y_2$  of  $L_\sigma$ . Hence we have  $\dim(\Sigma_{y_1} \cap \Sigma_{y_2}) \geq 1$ . On the other hand, both  $\Sigma_{y_1}$  and  $\Sigma_{y_2}$  are 2-dimensional maximal totally geodesic spheres of isotropy angle  $\frac{\pi}{6}$  (i.e., of principal type). Hence, according to Lemma 2, they coincide with each other. Thus  $\Sigma_y$ 's ( $y \in L_\sigma$ ) coincide with one another. This implies that  $L_\sigma$  is a one-point set, which contradicts that  $v$  is focal normal vector field. Therefore,  $\mathcal{FH}_{eK}$  is empty set, that is,  $M$  has no focal set. Hence  $\widetilde{M}$  also have no focal set. This fact implies that  $\widetilde{M}$  is totally geodesic. That is, we have  $T_{\tilde{0}}\widetilde{M} = \text{Ker } \widetilde{A}_w$  for any  $w \in T_{\tilde{0}}^\perp\widetilde{M}$ . Hence, from Lemma 6, we have  $T_{eK}M = \text{Ker } A_w \cap \text{Ker } R(\cdot, w)w$  for any  $w \in T_{eK}^\perp M$ . Since  $\Sigma_{eK}$  is of isotropy angle  $\frac{\pi}{6}$ ,  $\text{Ker } R(\cdot, w)w$  is a maximal abelian subspace of  $\mathfrak{p}$  containing  $w$ . Hence we have  $\dim M \leq \dim(\text{Ker } R(\cdot, w)w \ominus \text{Span}\{w\}) = 1$ , that is,  $\dim M = 1$ . Therefore we have  $\dim SU(3)/SO(3) = \dim M + \dim \Sigma_{eK} = 3$ . This contradicts  $\dim SU(3)/SO(3) = 5$ . Therefore there exists no equifocal submanifold with non-flat section in  $SU(3)/SO(3)$ .

(The case of  $SU(6)/Sp(3)$ ) Suppose that there exists an equifocal submanifold  $M$  with non-flat section in  $SU(6)/Sp(3)$ . Denote by  $r$  the codimension of  $M$ . Without loss of generality, we may assume that  $eK \in M$ . According to the result in [Koi3] stated in Introduction, the section  $\Sigma_{eK}$  is a totally geodesic  $r$ -dimensional sphere or an  $r$ -dimensional totally geodesic real projective space. First we consider the case where  $\Sigma_{eK}$  is a  $r$ -dimensional totally geodesic sphere. Take  $\sigma \in \mathfrak{FH}_{eK}$  and a focal normal vector field  $v$  of  $M$  such that  $p := \exp^\perp(v_{eK})$  belongs to  $\sigma$  but that it does not belong to members of  $\mathfrak{FH}_{eK}$  other than  $\sigma$ . Let

$F_\sigma$  and  $L_\sigma$  be as above. According to Table 2, there uniquely exists a maximal totally geodesic sphere containing  $\Sigma_y$  for each  $y \in L_\sigma$  and it is a 5-dimensional totally geodesic sphere of the isotropy angle  $\frac{\pi}{6}$  (hence of principal type). Denote by  $\tilde{\Sigma}_y$  this maximal totally geodesic sphere containing  $\Sigma_y$ . Then, for any distinct points  $y_1$  and  $y_2$  of  $L_\sigma$ , we have  $\Sigma_{y_1} \cap \Sigma_{y_2} = \sigma$  and hence  $\dim(\tilde{\Sigma}_{y_1} \cap \tilde{\Sigma}_{y_2}) \geq 1$ . On the other hand, both  $\tilde{\Sigma}_{y_1}$  and  $\tilde{\Sigma}_{y_2}$  are maximal totally geodesic spheres of principal type. Therefore, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_y$ 's ( $y \in L_\sigma$ ) coincide with one another. So we denote  $\tilde{\Sigma}_y$  by  $\tilde{\Sigma}_\sigma$ . For any  $\sigma_1, \sigma_2 \in \mathcal{FH}_{eK}$ , we have  $\dim(\tilde{\Sigma}_{\sigma_1} \cap \tilde{\Sigma}_{\sigma_2}) \geq \dim \Sigma_{eK} = r$ , and both  $\tilde{\Sigma}_{\sigma_1}$  and  $\tilde{\Sigma}_{\sigma_2}$  are totally geodesic spheres of principal type. Hence, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_\sigma$ 's ( $\sigma \in \mathcal{FH}_{eK}$ ) coincide with one another. So we denote  $\tilde{\Sigma}_\sigma$  ( $\sigma \in \mathcal{FH}_{eK}$ ) by  $\tilde{\Sigma}$ . Since  $\bigcup_{\sigma \in \mathcal{FH}_{eK}} L_\sigma \subset \tilde{\Sigma}$ , we have

$\dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_\sigma \right) \leq \dim \tilde{\Sigma} = 5$ . On the other hand, since  $\Sigma_{eK}$  is of isotropy angle  $\frac{\pi}{6}$ ,  $\text{Ker } R(\cdot, w)w$  is a maximal abelian subspace of  $\mathfrak{p}$  containing  $w$  for each  $w \in \Sigma_{eK}$ . Hence we have

$$\dim M \leq \dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_\sigma + (\text{Ker } R(\cdot, w)w \ominus \text{Span}\{w\}) \right) = 6.$$

Therefore we have  $\dim SU(6)/Sp(3) = \dim M + \dim \Sigma_{eK} \leq 11$ . This contradicts  $\dim SU(6)/Sp(3) = 14$ . Next we consider the case where  $\Sigma_{eK}$  is a  $r$ -dimensional totally geodesic real projective space. Let  $\sigma, v, F_\sigma$  and  $L_\sigma$  be as above. According to Table 2, there uniquely exists a maximal totally geodesic real projective space containing  $\Sigma_y$  for each  $y \in L_\sigma$  and it is a 3-dimensional totally geodesic real projective space of the isotropy angle  $\frac{\pi}{6}$  (hence of principal type). Denote by  $\tilde{\Sigma}_y$  this maximal totally geodesic sphere containing  $\Sigma_y$ . For any distinct points  $y_1$  and  $y_2$  of  $L_\sigma$ , we have  $\dim(\tilde{\Sigma}_{y_1} \cap \tilde{\Sigma}_{y_2}) \geq 1$ . On the other hand, both  $\tilde{\Sigma}_{y_1}$  and  $\tilde{\Sigma}_{y_2}$  are maximal totally geodesic real projective spaces of principal type. Therefore, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_y$ 's ( $y \in L_\sigma$ ) coincide with one another. So we denote  $\tilde{\Sigma}_y$  by  $\tilde{\Sigma}_\sigma$ . For any  $\sigma_1, \sigma_2 \in \mathcal{FH}_{eK}$ , we have  $\dim(\tilde{\Sigma}_{\sigma_1} \cap \tilde{\Sigma}_{\sigma_2}) \geq \dim \Sigma_{eK} = r$ , and both  $\tilde{\Sigma}_{\sigma_1}$  and  $\tilde{\Sigma}_{\sigma_2}$  are totally geodesic real projective space of principal type. Hence, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_\sigma$ 's ( $\sigma \in \mathcal{FH}_{eK}$ ) coincide with one another. So we denote  $\tilde{\Sigma}_\sigma$  ( $\sigma \in \mathcal{FH}_{eK}$ ) by  $\tilde{\Sigma}$ . Since  $\bigcup_{\sigma \in \mathcal{FH}_{eK}} L_\sigma \subset \tilde{\Sigma}$ , we have

$\dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_\sigma \right) \leq \dim \tilde{\Sigma} = 3$ . On the other hand, since  $\Sigma_{eK}$  is of isotropy angle  $\frac{\pi}{6}$ ,  $\text{Ker } R(\cdot, w)w$  is a maximal abelian subspace of  $\mathfrak{p}$  containing  $w$  for each

$w \in \Sigma_{eK}$ . Hence we have

$$\dim M \leq \dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_{\sigma} + (\text{Ker } R(\cdot, w)w \ominus \text{Span}\{w\}) \right) = 4.$$

Therefore we have  $\dim SU(6)/Sp(3) = \dim M + \dim \Sigma_{eK} \leq 7$ . This contradicts  $\dim SU(6)/Sp(3) = 14$ . Therefore it follows that there exists no equifocal submanifold with non-flat section in  $SU(6)/Sp(3)$ .

(The case of  $E_6/F_4$ ) According to Table 8, the only maximal totally geodesic sphere of dimension greater than one in  $E_6/F_4$  is of 9-dimensional and isotropy angle  $\frac{\pi}{6}$ , and the only maximal totally geodesic real projective space of greater than one in  $E_6/F_4$  is of 3-dimensional and isotropy angle  $\frac{\pi}{6}$ . By noticing these facts and discussing similarly, we can show that there exists no equifocal submanifold with non-flat section in  $E_6/F_4$ .

(The case of  $SU(3)$ ) According to Table 10, the only maximal totally geodesic sphere of dimension greater than one in  $SU(3)$  is of 3-dimensional and isotropy angle  $\frac{\pi}{6}$ , and the only maximal totally geodesic real projective space of greater than one in  $SU(3)$  is of 3-dimensional and isotropy angle  $\frac{\pi}{6}$ . By noticing these facts and discussing similarly, we can show that there exists no equifocal submanifold with non-flat section in  $SU(3)$ . q.e.d.

Next we prove Theorem B.

*Proof of Theorem B.* (The case of  $SU(2+q)/S(U(2) \times U(q))$ ) Let  $M$  be an equifocal submanifold with non-flat section in  $SU(6)/Sp(3)$  and  $\Sigma_x$  be the section of  $M$  through  $x(\in M)$ . Denote by  $r$  the codimension of  $M$ . Without loss of generality, we may assume that  $eK \in M$ . According to the result in [Koi3] stated in Introduction, the section  $\Sigma_{eK}$  is a  $r$ -dimensional totally geodesic sphere or a  $r$ -dimensional totally geodesic real projective space. Suppose that  $\Sigma_{eK}$  is of principal type. First we consider the case where  $\Sigma_{eK}$  is a  $r$ -dimensional totally geodesic sphere. Take  $\sigma \in \mathfrak{FH}_{eK}$  and a focal normal vector field  $v$  of  $M$  such that  $p := \exp^{\perp}(v_{eK})$  belongs to  $\sigma$  but that it does not belong to members of  $\mathfrak{FH}_{eK}$  other than  $\sigma$ . Let  $F_{\sigma}$  and  $L_{\sigma}$  be as in the proof of Theorem A. According to Table 3, there uniquely exists a maximal totally geodesic sphere containing  $\Sigma_y$  for each  $y \in L_{\sigma}$ . Denote by  $\tilde{\Sigma}_y$  this maximal totally geodesic sphere. Also, according to the table, if  $q = 2$ , then it is a 2-dimensional totally geodesic sphere of the isotropy angle  $\arctan \frac{1}{3}$  (hence of principal type), if  $q = 3$ , then it is a 2-dimensional totally geodesic sphere of the isotropy angle  $\arctan \frac{1}{3}$  (hence of principal type) or a 4-dimensional totally geodesic sphere of the isotropy angle  $\arctan \frac{1}{2}$  (hence of principal type) and, if  $q \geq 4$ , then it is a 2-dimensional totally geodesic sphere of the isotropy angle  $\arctan \frac{1}{3}$  (hence of principal type) or  $\arctan \frac{1}{2}$  (hence of principal type). For any distinct points  $y_1$  and  $y_2$  of  $L_{\sigma}$ , we have

$\dim(\tilde{\Sigma}_{y_1} \cap \tilde{\Sigma}_{y_2}) \geq \dim \sigma \geq 1$  (hence  $\tilde{\Sigma}_{y_1}$  and  $\tilde{\Sigma}_{y_2}$  have the same isotropy angle). Therefore, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_y$ 's ( $y \in L_\sigma$ ) coincide with one another. So we denote  $\tilde{\Sigma}_y$  by  $\tilde{\Sigma}_\sigma$ . For any  $\sigma_1, \sigma_2 \in \mathcal{FH}_{eK}$ , we have  $\dim(\tilde{\Sigma}_{\sigma_1} \cap \tilde{\Sigma}_{\sigma_2}) \geq \dim \Sigma_{eK} = r$ . Therefore, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_\sigma$ 's ( $\sigma \in \mathcal{FH}_{eK}$ ) coincide with one another. So we denote  $\tilde{\Sigma}_\sigma$  ( $\sigma \in \mathcal{FH}_{eK}$ ) by  $\tilde{\Sigma}$ . Since  $\bigcup_{\sigma \in \mathcal{FH}_{eK}} L_\sigma \subset \tilde{\Sigma}$ , we have

$$\dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_\sigma \right) \leq \dim \tilde{\Sigma} = \begin{cases} 2 & (\text{when } q = 2 \text{ or } q \geq 4) \\ 4 & (\text{when } q = 3) \end{cases}$$

On the other hand, since  $\Sigma_{eK}$  is of isotropy angle  $\arctan \frac{1}{3}$  or  $\arctan \frac{1}{2}$ ,  $\text{Ker } R(\cdot, w)w$  is a maximal abelian subspace of  $\mathfrak{p}$  containing  $w$  for each  $w \in \Sigma_{eK}$ . Hence we have

$$\begin{aligned} \dim M &\leq \dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_\sigma + (\text{Ker } R(\cdot, w)w \ominus \text{Span}\{w\}) \right) \\ &\leq \begin{cases} 3 & (\text{when } q = 2 \text{ or } q \geq 4) \\ 5 & (\text{when } q = 3) \end{cases} \end{aligned}$$

Therefore we have

$$\dim SU(2+q)/S(U(2) \times U(q)) = \dim M + \dim \Sigma_{eK} \leq \begin{cases} 5 & (\text{when } q = 2 \text{ or } q \geq 4) \\ 9 & (\text{when } q = 3) \end{cases}$$

This contradicts  $\dim SU(2+q)/S(U(2) \times U(q)) = 4q$ . Next we consider the case where  $\Sigma_{eK}$  is a  $r$ -dimensional totally geodesic real projective space. Take  $\sigma, v, F_\sigma$  and  $L_\sigma$  be as above. According to Table 3, we have  $q \neq 2$  and there uniquely exists a maximal totally geodesic real projective space containing  $\Sigma_y$  for each  $y \in L_\sigma$  when  $q \geq 3$ . Denote by  $\tilde{\Sigma}_y$  this maximal totally geodesic real projective space. Also, according to the table, it is a 2-dimensional totally geodesic real projective space of isotropy angle  $\arctan \frac{1}{2}$ . For any distinct points  $y_1$  and  $y_2$  of  $L_\sigma$ , we have  $\dim(\tilde{\Sigma}_{y_1} \cap \tilde{\Sigma}_{y_2}) \geq \dim \sigma \geq 1$ . Therefore, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_y$ 's ( $y \in L_\sigma$ ) coincide with one another. So we denote  $\tilde{\Sigma}_y$  by  $\tilde{\Sigma}_\sigma$ . For any  $\sigma_1, \sigma_2 \in \mathcal{FH}_{eK}$ , we have  $\dim(\tilde{\Sigma}_{\sigma_1} \cap \tilde{\Sigma}_{\sigma_2}) \geq \dim \Sigma_{eK} = r$ . Therefore, according to Lemma 2, they coincide with each other. Thus  $\tilde{\Sigma}_\sigma$ 's ( $\sigma \in \mathcal{FH}_{eK}$ ) coincide with one another. So we denote  $\tilde{\Sigma}_\sigma$  ( $\sigma \in \mathcal{FH}_{eK}$ ) by  $\tilde{\Sigma}$ .

Since  $\bigcup_{\sigma \in \mathcal{FH}_{eK}} L_\sigma \subset \tilde{\Sigma}$ , we have  $\dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_\sigma \right) \leq \dim \tilde{\Sigma} = 2$ . On the other hand, since  $\Sigma_{eK}$  is of isotropy angle  $\arctan \frac{1}{2}$ ,  $\text{Ker } R(\cdot, w)w$  is a maximal abelian

subspace of  $\mathfrak{p}$  containing  $w$  for each  $w \in \Sigma_{eK}$ . Hence we have

$$\dim M \leq \dim \left( \sum_{\sigma \in \mathcal{FH}_{eK}} T_{eK} L_{\sigma} + (\text{Ker } R(\cdot, w)w \ominus \text{Span}\{w\}) \right) \leq 3.$$

Therefore we have  $\dim SU(2+q)/S(U(2) \times U(q)) = \dim M + \dim \Sigma_{eK} \leq 5$ . This contradicts  $\dim SU(2+q)/S(U(2) \times U(q)) = 4q$ . Therefore  $\Sigma_{eK}$  is of singular type. This completes the proof in case of  $G/K = SU(2+q)/S(U(2) \times U(q))$ .

(The case of  $SO(2+q)/SO(2) \times SO(q)$ ) According to Table 4, there exists the only maximal totally geodesic sphere of principal type and dimension greater than one in  $SO(2+q)/SO(2) \times SO(q)$  and it is of dimension two and isotropy angle  $\arctan \frac{1}{2}$ , and there exists no totally geodesic real projective space of principal type in  $SO(2+q)/SO(2) \times SO(q)$ . By noticing these facts and discussing similarly, we can show that the sections of equifocal submanifolds with non-flat section in  $SO(2+q)/SO(2) \times SO(q)$  are of singular type.

(The case of  $SO(10)/U(5)$ ) According to Table 5, there exists no totally geodesic rank one symmetric space of principal type and dimension greater than one in  $SO(10)/U(5)$ . Hence it follows directly that the sections of equifocal submanifolds with non-flat section in  $SO(10)/U(5)$  are of singular type.

(The case of  $Sp(2+q)/Sp(2) \times Sp(q)$ ) According to Table 6, if  $q = 2$ , then a maximal totally geodesic sphere of principal type and dimension greater than one is a 3-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{3}$  or a 4-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{2}$  and there exists no maximal totally geodesic real projective space of principal type and dimension greater than one. Also, if  $q = 3, 4$ , then a maximal totally geodesic sphere of principal type and dimension greater than one is a 3-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{3}$  or a 4-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{2}$  and a maximal totally geodesic real projective space of principal type and dimension greater than one is a 2-dimensional totally geodesic real projective space of isotropy angle  $\arctan \frac{1}{2}$ . Also, if  $q \geq 5$ , then a maximal totally geodesic sphere of principal type and dimension greater than one is a 3-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{3}$  and a maximal totally geodesic real projective space of principal type and dimension greater than one is a 2-dimensional totally geodesic real projective space of isotropy angle  $\arctan \frac{1}{2}$ . By noticing these facts and discussing similarly, we can show that the sections of equifocal submanifolds with non-flat section in  $Sp(2+q)/Sp(2) \times Sp(q)$  are of singular type.

(The case of  $E_6/Spin(10) \cdot U(1)$ ) According to Table 7, there exists no totally geodesic rank one symmetric space of principal type and dimension greater than one in  $E_6/Spin(10) \cdot U(1)$ . Hence it follows directly that the sections of equifocal submanifolds with non-flat section in  $E_6/Spin(10) \cdot U(1)$  are of singular type.



(The case of  $G_2/SO(4)$ ) According to Table 9, the only maximal totally geodesic rank one symmetric space of principal type and dimension greater than one is a 3-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{3\sqrt{3}}$ . By noticing this fact and discussing similarly, we can show that the sections of equifocal submanifolds with non-flat section in  $G_2/SO(4)$  are of singular type.

(The case of  $Sp(2)$ ) According to Table 11, the only maximal totally geodesic rank one symmetric space of principal type and dimension greater than one is a 3-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{3}$ . By noticing this fact and discussing similarly, we can show that the sections of equifocal submanifolds with non-flat section in  $Sp(2)$  are of singular type.

(The case of  $G_2$ ) According to Table 12, the only maximal totally geodesic rank one symmetric space of principal type and dimension greater than one is a 3-dimensional totally geodesic sphere of isotropy angle  $\arctan \frac{1}{3\sqrt{3}}$ . By noticing this fact and discussing similarly, we can show that the sections of equifocal submanifolds with non-flat section in  $G_2$  are of singular type. q.e.d.

## References

- [A] M. M. Alexandrino, Singular riemannian foliations with sections, Illinois J. Math. **48** (2004) 1163-1182.
- [AG] M. M. Alexandrino and C. Gorodski, Singular Riemannian foliations with sections, transnormal maps and basic forms, arXiv:math.GT/0608069.
- [AT] M. M. Alexandrino and D. Töben, Singular Riemannian foliations on simply connected spaces, Differential Geom. Appl. **24** (2006) 383-397.
- [BCO] J. Berndt, S. Console and C. Olmos, Submanifolds and holonomy, Research Notes in Mathematics 434, CHAPMAN & HALL/CRC Press, Boca Raton, London, New York Washington, 2003.
- [Bi] L. Biliotti, Coisotropic and polar actions on compact irreducible Hermitian symmetric spaces, Trans. Amer. Math. Soc. **358** (2006) 3003-3022.
- [BS] R. Bott and H. Samelson, Applications of the theory of Morse to symmetric spaces, Amer. J. Math. **80** (1958) 964-1029. Correction in Amer. J. Math. **83** (1961) 207-208.
- [Br] M. Brück, Equifocal families in symmetric spaces of compact type, J. reine angew. Math. **515** (1999), 73-95.
- [Ch] U. Christ, Homogeneity of equifocal submanifolds, J. Differential Geometry **62** (2002) 1-15.
- [CN] B.Y. Chen and T. Nagano, Totally geodesic submanifolds of symmetric spaces II, Duke Math. J. **45** (1978) 405-425.
- [D] J. Dadok, Polar coordinates induced by actions of compact Lie groups, Trans. Amer. Math. Soc. **228** (1985) 125-137.
- [E] H. Ewert, Equifocal submanifolds in Riemannian symmetric spaces, Doctoral thesis.
- [H] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, 1978.

- [HLO] E. Heintze, X. Liu and C. Olmos, Isoparametric submanifolds and a Chevalley-type restriction theorem, Integrable systems, geometry, and topology, 151-190, AMS/IP Stud. Adv. Math. 36, Amer. Math. Soc., Providence, RI, 2006.
- [HPTT] E. Heintze, R.S. Palais, C.L. Terng and G. Thorbergsson, Hyperpolar actions on symmetric spaces, Geometry, topology and physics for Raoul Bott (ed. S. T. Yau), Conf. Proc. Lecture Notes Geom. Topology **4**, Internat. Press, Cambridge, MA, 1995 pp214-245.
- [Kl1] S. Klein, Totally geodesic submanifolds of the complex quadric, Differential Geom. Appl., **26** (2008) 79-96.
- [Kl2] S. Klein, Totally geodesic submanifolds of the complex and the quaternionic 2-Grassmannians, Trans. Amer. Math. Soc., **361** (2009) 4927-4967.
- [Kl3] S. Klein, Reconstructing the geometric structure of a Riemannian symmetric space from its Satake diagram, Geom. Dedicata, **138** (2009) 25-50.
- [Kl4] S. Klein, Totally geodesic submanifolds of the exceptional Riemannian symmetric spaces of rank 2, arXiv:math.DG/0809.1319v1.
- [Koi1] N. Koike, Tubes of non-constant radius in symmetric spaces, Kyushu J. Math. **56** (2002) 267-291.
- [Koi2] N. Koike, On proper Fredholm submanifolds in a Hilbert space arising from submanifolds in a symmetric space, Japan. J. Math. **28** (2002) 61-80.
- [Koi3] N. Koike, On non-existenceness of equifocal submanifolds with non-flat section, Asian J. Math. **12** (2008) 421-442.
- [Kol1] A. Kollross, A Classification of hyperpolar and cohomogeneity one actions, Trans. Amer. Math. Soc. **354** (2001) 571-612.
- [Kol2] A. Kollross, Polar actions on symmetric spaces, J. Differential Geometry **77** (2007) 425-482.
- [Pa] R.S. Palais, Morse theory on Hilbert manifolds, Topology **2** (1963) 299-340.
- [PaTe] R.S. Palais and C.L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. **1353**, Springer, Berlin, 1988.
- [PoTh1] F. Podestá and G. Thorbergsson, Polar actions on rank-one symmetric spaces, J. Differential Geometry **53** (1999) 131-175.
- [PoTh2] F. Podestá and G. Thorbergsson, Polar and coisotropic actions on Kähler manifolds, Trans. Amer. Math. Soc. **354** (2002) 1759-1781.
- [TeTh] C.L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, J. Differential Geometry **42** (1995) 665-718.
- [Th] G. Thorbergsson, Isoparametric foliations and their buildings, Ann. of Math. **133** (1991) 429-446.
- [Tö] D. Töben, Parallel focal structure and singular Riemannian foliations, Trans. Amer. Math. Soc. **358** (2006) 1677-1704.

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